Lezione 2

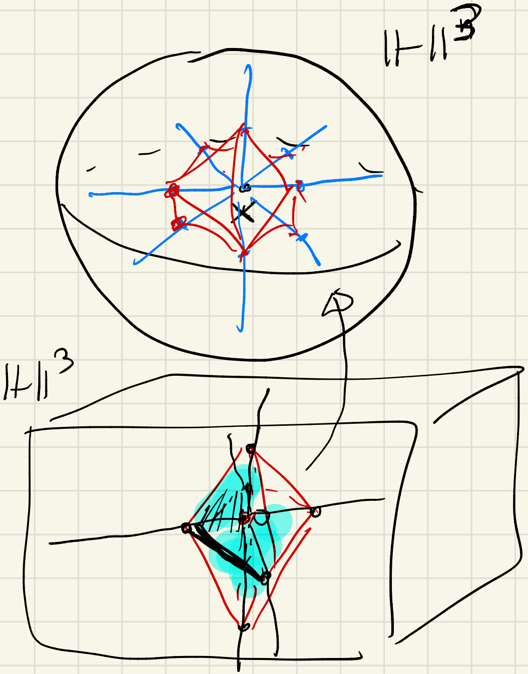
P solido platonico $\quad P \subseteq \mathbb{R}^3$

$t > 0$

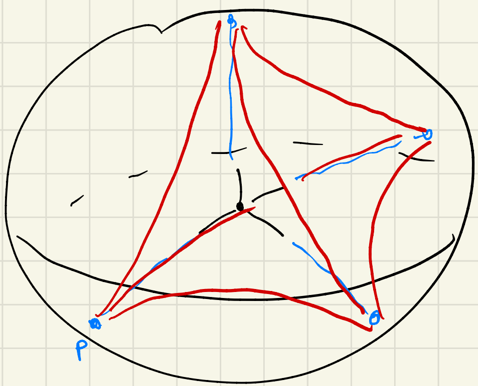
$$\mathrm{Conv}\left(\exp\left(t(x_1, \ldots, x_k)\right)\right) = P(-t)$$

$P(-t)$ ha tutte le simmetrie di P

$P(-\infty)$ i vertici sono tutti ideali



$\mathbb{R}^3 \underset{\text{isom.}}{\cong} T_x \mathbb{H}^3$

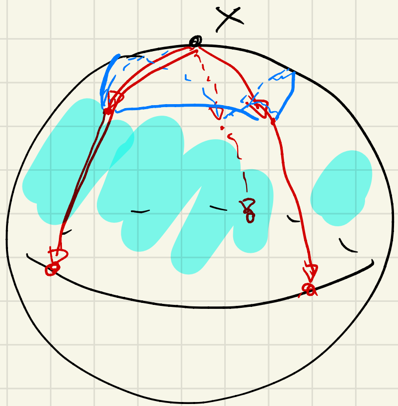


TETRAEDRO IDEALE REGOLARE

$P(0) = P$

$P(t) =$ versione in S^3

$0 < t \leq \frac{\pi}{2}$



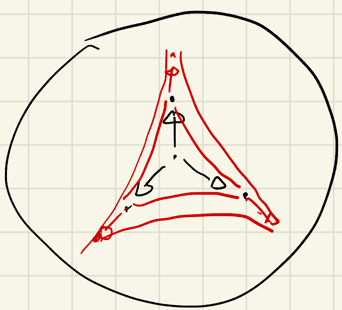
n valenza dei vertici di P

$n = 3, 4, 5$

$t \to \frac{\pi}{2}$ $\quad P(t) \to$ EMISFERO NORD (METÀ SFERA)

$P(t) \quad \forall t \in [-\infty, \frac{\pi}{2}]$

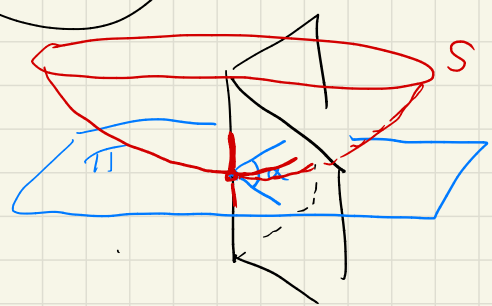
$\vartheta(t)$ angolo diedrale $\quad \vartheta(t)$ strettamente crescente



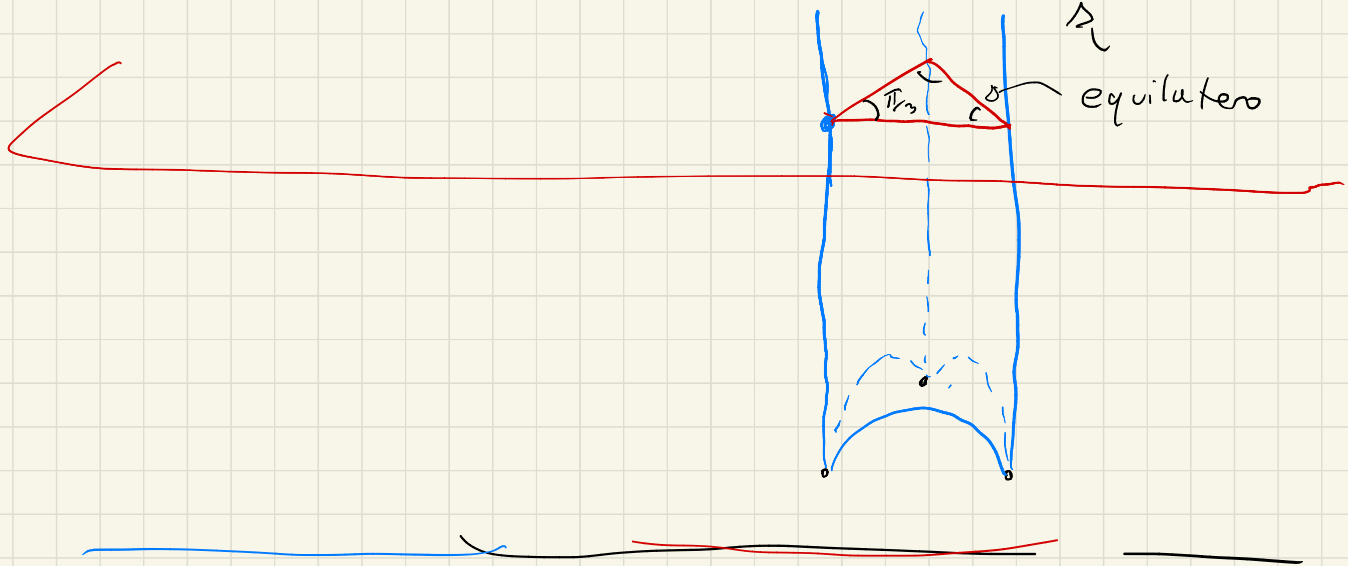
<u>Prop</u>: $\vartheta\left([-\infty, \frac{\pi}{2}]\right) = \left[\frac{n-2}{n}\pi, \pi\right]$

<u>dim</u> $\quad \vartheta\left(\frac{\pi}{2}\right) = \pi \quad P\left(\frac{\pi}{2}\right) =$ semisfera

$\vartheta(-\infty)$ POLIEDRO IDEALE $= \pi \frac{n-2}{n}$



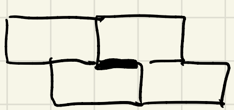
<u>Def</u>: con piano π



TASSELLAZIONE

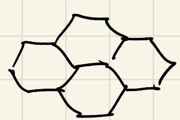
Def: Una TASSELLAZIONE di $X^n = \mathbb{H}^n, \mathbb{R}^n, S^n$ è una collezione di poliedri che

1) ricoprono X^n
2) si intersecano in facce comuni



NO

$\mathbb{R}^2$



SI

Esercizio: $0 \leq \alpha, \beta, \gamma \leq \pi/2$ $\exists!$ triangolo T con questi angoli in $\mathbb{H}^2, \mathbb{R}^2, S^2$ se $\alpha+\beta+\gamma <, =, > \pi$

(unico a meno di isom + simil. $\mathbb{R}^2$)

$a, b, c \geq 2$

$\Delta(a,b,c)$ = triangolo con angoli $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$

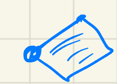
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < 1$ $\mathbb{H}^2$

$= 1$ $(3,3,3)$ $(2,4,4)$ $(2,3,6)$

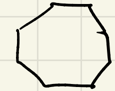
> 1 $(2,2,c)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$

Se rifletto Δ iterativamente ottengo tassellazione di $\mathbb{H}^2$, $\mathbb{R}^2$, S^2

Schläfli NOTAZIONE : Tassellazione di X^n è REGOLARE se le isometrie della tass. agiscono transitivamente sui FLAG $:= f_0 \subseteq f_1 \subseteq \dots \subseteq f_n$ $\dim f_i = i$



Tassellazione regolare in S^n $\rightsquigarrow$ Poliedro convesso regolare in $\mathbb{R}^{n+1}$

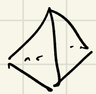
• $\{n\}$:= poligono reg. con n lati ⟶  tass. di S^1

n=8

$\{\infty\}$ = tassel. di $\mathbb{R}$ in ∞ 

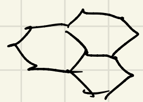
•• $\{p,q\}$ = tassellazione di p-goni regolari che si incontrano a q su ogni vertice in $\mathbb{H}^2, \mathbb{R}^2, S^2$

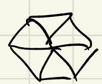
$p,q \geq 3$

$\{3,3\}$  $\{3,4\}$  $\{3,5\}$ icos

S^2

$\{4,3\}$  $\{5,3\}$ dod

$\{6,3\}$  $\mathbb{R}^2$ $\{4,4\}$ 

$\{3,6\}$  $\mathbb{R}^2$ tutte le altre in $\mathbb{H}^2$

$\{p,\infty\}$ ←

$\{\infty,p\}$ ← $\mathbb{H}^2$

$\{\infty,\infty\}$ ←

••• $\{p,q,r\}$ tassellazione di X^3

con poliedri $\{p,q\}$ che si incontrano r volte su ogni spigolo

S^3 $\{3,3,3\}$ (TET), $\{3,3,4\}$ (CUBO), $\{3,3,5\}$ (600), $\{3,4,3\}$ (24-celle), $\{4,3,3\}$ (IPER OTTAEDR), $\{5,3,3\}$ (120)

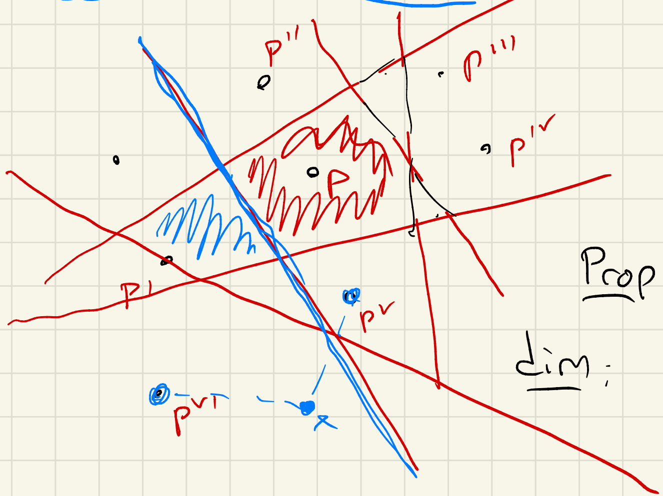
$\mathbb{R}^3$ $\{4,3,4\}$

$\mathbb{H}^3$ $\{3,3,6\}$

 $\frac{\pi}{3}$

Duale di $\{a, b, \dots, z\}$ è $\{z, \dots, b, a\}$

Tassellazioni di Voronoi:



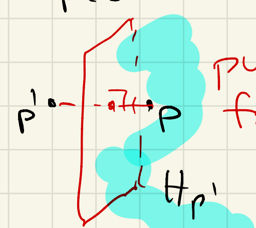
$S \subseteq \mathbb{X}^n$ discreto chiuso $\rightsquigarrow$ tassellazione di $\mathbb{X}^n$

$p \in S$ $D(p) = \{x \in \mathbb{X}^n \mid d(x,p) \leq d(x,p') \ \forall p' \in S\}$

Prop: $\{D(p)\}_{p \in S}$ tassellazione di $\mathbb{X}^n$

dim:

Ex:



punti equidistanti formano iperpiano

$D(p) = \bigcap_{\substack{p' \in S \\ p' \neq p}} H_{p'}$ $\quad$ $H_{p'}$ semispazio $\quad$ $\bigcup_{p \in S} D(p) = \mathbb{X}^n$

↑ POLIEDRO

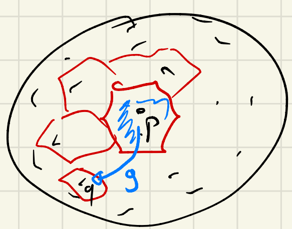
LOC. FINITA perché S denso

$p, p' \in S$ $\quad D(p) \cap D(p') = D(p) \cap \partial H_{p'}$ è $\emptyset$ o faccia $\square$

<u>Def</u>: $\Gamma < \mathrm{Isom}(\mathbb{X}^n)$ discreto

Un DOMINIO FONDAMENTALE per Γ è un poliedro $P \subseteq \mathbb{X}^n$ t.c. $\{g(P)\}_{g \in \Gamma}$ formano tassellazione

<u>Def</u>: Un DOMINIO DI DIRICHLET è un dominio ottenuto dalla costruzione di Voronoi prendendo $S = \Gamma(p)$ per qualche $p \in \mathbb{X}^n$ con Γ_p banale



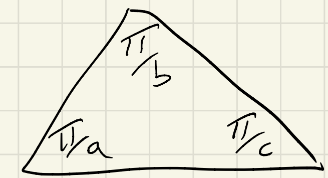
TASSELLAZIONE

DIRICHLET $\Rightarrow$ FONDAMENTALE

Γ agisce su questa tassellazione in modo libero e transitivo

Poliedri di Coxeter: $P \subseteq \mathbb{X}^n$ POLIEDRO è DI COXETER

se i suoi angoli diedrali dividono sempre π



$F_1, F_2, \dots, F_i, \dots$

a_{ij} = angolo fra F_i e F_j quando si intersecano

$r_1, r_2, \dots, r_i, \dots$ $\quad r_i \in \mathrm{Isom}(\mathbb{X}^n)$ riflessione lungo F_i

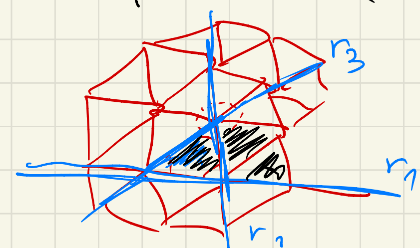
Teo: Specchiando P iterativamente si ottiene una tassellazione di $\mathbb{X}^n$ su cui $\Gamma = \langle r_1, \dots, r_i, \dots \rangle$ agisce libero e transitivo

Quindi Γ discreto e P è un suo dominio fondamentale

$P \rightsquigarrow \Gamma \rhd \Gamma'$ senza torsione (i.f. $= k$, Selberg)

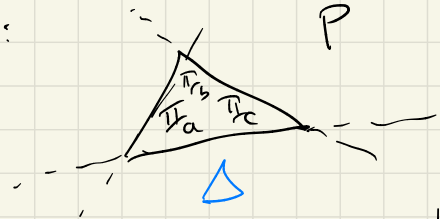
$\mathbb{X}^n / \Gamma' = M$ varietà ip.

M si tassella in k copie di P

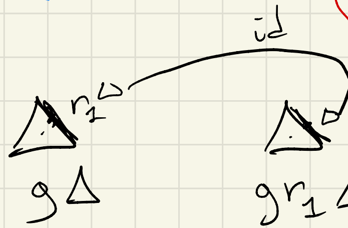


Γ

dim:



$\Gamma = \langle r_1, r_2, r_3 \rangle$



$\{g\Delta\}_{g\in\Gamma} \big/ \sim \; = S$ connessa sup. iperbolica completa

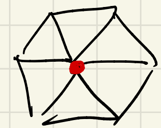
$\downarrow$ isom. locale

$\mathbb{H}^2$

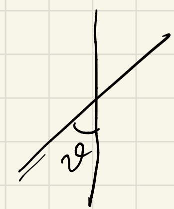
rivestimento

$\Downarrow$

isometria



$$\Gamma = \langle r_1, \ldots, r_k, \ldots \mid (r_i r_j)^{a_{ij}}, r_i^2 \rangle$$



$\square$